

Schrödinger operators with random δ magnetic fields

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Abstract. We shall consider the Schrödinger operators on \mathbb{R}^2 with random δ magnetic fields. Under some mild conditions on the positions and the fluxes of the δ -fields, we prove the spectrum coincides with $[0, \infty)$ and the integrated density of states (IDS) decays exponentially at the bottom of the spectrum (Lifshitz tail), by using the Hardy type inequality by Laptev-Weidl [21]. We also give a lower bound for IDS at the bottom of the spectrum.

1 Introduction

We consider the Schrödinger operators on \mathbb{R}^2 with random magnetic fields

$$\mathcal{L}_\omega = \left(\frac{1}{i} \nabla - \mathbf{a}_\omega \right)^2,$$

where ω is an element from some probability space (Ω, \mathbf{P}) and the vector-valued function $\mathbf{a}_\omega(x) = (a_{\omega,1}(x), a_{\omega,2}(x))$ ($x = (x_1, x_2) \in \mathbb{R}^2$) is the magnetic vector potential dependent on ω . The magnetic field corresponding to \mathbf{a}_ω is given by $\text{curl } \mathbf{a}_\omega = \partial_1 a_{\omega,2} - \partial_2 a_{\omega,1}$ ($\partial_j = \partial/\partial x_j$) and we assume

$$\text{curl } \mathbf{a}_\omega = \sum_{\gamma \in \Gamma_\omega} 2\pi \alpha_\gamma(\omega) \delta_\gamma \quad (1)$$

in the distribution sense, where Γ_ω is a discrete set in \mathbb{R}^2 dependent on ω without accumulation points in \mathbb{R}^2 , $\alpha(\omega) = \{\alpha_\gamma(\omega)\}_{\gamma \in \Gamma_\omega}$ is a sequence of real numbers satisfying $0 \leq \alpha_\gamma(\omega) < 1$ and dependent on ω , and δ_γ is the Dirac measure supported on the point γ . For any given $(\Gamma_\omega, \alpha(\omega))$, we can construct vector potential $\mathbf{a}_\omega \in C^\infty(\mathbb{R}^2 \setminus \Gamma_\omega; \mathbb{R}^2)$ satisfying (1) (see (7) below). The assumption $0 \leq \alpha_\gamma(\omega) < 1$ loses no generality, because we can shift the value of α_γ by any integer value by using suitable gauge transform (see Lemma 2.1 below).

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Before stating our assumptions, we prepare some notation used in the present paper. For $S \subset \mathbb{R}^2$, $x \in \mathbb{R}^2$, and $r > 0$, let $S + x = \{s + x \mid s \in S\}$ and $rS = \{rs \mid s \in S\}$. For $k \geq 0$, let

$$Q_k = \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid -k - \frac{1}{2} \leq x_j < k + \frac{1}{2} \ (j = 1, 2) \right\},$$

which is a square with edge length $2k + 1$ centered at the origin. Especially Q_0 is a unit square centered at the origin. The boundary of a set S is denoted by ∂S . The open disc of radius r centered at x is denoted by $B_x(r)$, that is,

$$B_x(r) = \{y \in \mathbb{R}^2 \mid |y - x| < r\}.$$

Our assumption is as follows.

Assumption 1.1. *Let (Ω, \mathbf{P}) be a probability space, Γ_ω a discrete set in \mathbb{R}^2 dependent on $\omega \in \Omega$ without accumulation points in \mathbb{R}^2 , and $\alpha(\omega) = \{\alpha_\gamma(\omega)\}_{\gamma \in \Gamma_\omega}$ a sequence of real numbers with $0 \leq \alpha_\gamma(\omega) < 1$ dependent on $\omega \in \Omega$. For a Borel set E in \mathbb{R}^2 , put*

$$\Phi_\omega(E) = \sum_{\gamma \in \Gamma_\omega \cap E} \alpha_\gamma(\omega).$$

We assume the following conditions (i)-(vi).

- (i) *For any Borel set E in \mathbb{R}^2 , the random variable $\Phi(E) : \omega \mapsto \Phi_\omega(E)$ is measurable with respect to $\omega \in \Omega$.*
- (ii) *For any finite distinct points $\{n_j\}_{j=1}^J$ with $n_j \in \mathbb{Z}^2$, and for any Borel sets $\{E_j\}_{j=1}^J$ with $E_j \subset n_j + Q_0$, the random variables $\{\Phi(E_j)\}_{j=1}^J$ are independent.*
- (iii) *For any Borel set $E \subset Q_0$, the random variables $\{\Phi(E + n)\}_{n \in \mathbb{Z}^2}$ are identically distributed.*
- (iv) *The mathematical expectation $\mathbf{E}[\Phi(Q_0)]$ is positive and finite. The variance $\mathbf{V}[\Phi(Q_0)]$ is finite.*
- (v) *$\Phi_\omega(\partial Q_0) = 0$ almost surely.*
- (vi) *One of the following two conditions (a) or (b) holds.*

- (a) *There exists a positive constant c with $0 < c \leq 1$ independent of ω such that the probability of the event ‘the following two conditions (2) and (3) simultaneously hold’ is positive for any $\epsilon > 0$.*

$$\Phi_\omega(Q_0) = \sum_{\gamma \in \Gamma_\omega \cap Q_0} \alpha_\gamma < \epsilon, \quad (2)$$

$$B_\gamma(c\sqrt{\alpha_\gamma}) \cap B_{\gamma'}(c\sqrt{\alpha_{\gamma'}}) = \emptyset, \quad B_\gamma(c\sqrt{\alpha_\gamma}) \cap \partial Q_0 = \emptyset$$

for every $\gamma, \gamma' \in \Gamma_\omega \cap Q_0$ with $\gamma \neq \gamma'$. (3)

- (b) *The probability of the event*

$$\sum_{\gamma \in \Gamma_\omega \cap Q_0} \sqrt{\alpha_\gamma} < \epsilon \quad (4)$$

is positive for any $\epsilon > 0$.

In a part of our main result, we assume a stronger condition as follows.

Assumption 1.2. *In addition to Assumption 1.1, there exist positive constants c_1 and δ_1 such that*

$$\mathbf{P} \{ (2) \text{ and } (3) \text{ hold} \} \geq c_1 \epsilon^{\delta_1} \quad (\text{if } (vi)(a) \text{ holds}), \quad (5)$$

$$\mathbf{P} \{ (4) \text{ holds} \} \geq c_1 \epsilon^{\delta_1} \quad (\text{if } (vi)(b) \text{ holds}) \quad (6)$$

for sufficiently small $\epsilon > 0$, where ϵ is the one in (2) or (4), respectively.

The assumption (2) means the magnetic flux through Q_0 can be arbitrarily small, and (3) means the points Γ_ω are separated farther than a constant multiple of the magnetic length $\sqrt{\alpha_\gamma}$ as the flux tends to 0. The assumption (4) is independent of the positions of the points Γ_ω , but the restriction on the flux is stronger than (2), since $0 \leq \alpha_\gamma \leq \sqrt{\alpha_\gamma} \leq 1$. If the number of $\Gamma_\omega \cap Q_0$ is bounded by a constant independent of ω , then (2) implies (4) by the Schwarz inequality. These conditions guarantee the spectrum of our Hamiltonian is $[0, \infty)$ (Theorem 1.3).

There are numerous examples satisfying Assumption 1.1 or 1.2. We list some typical examples below.

- (i) **Perturbation of a lattice.** Let $\Gamma_\omega = \{n + f_n(\omega)\}_{n \in \mathbb{Z}^2}$, where $\{f_n\}$ are independently, identically distributed (*abbrev.* i.i.d.) \mathbb{R}^2 -valued random

variables with values in the interior of Q_0 . The fluxes $\{\alpha_\gamma\}$ are $[0, 1)$ -valued i.i.d. random variables independent of $\{f_n\}$, satisfying $\mathbf{E}[\alpha_\gamma] > 0$ and

$$\mathbf{P}\{\alpha_\gamma < \epsilon\} > 0 \quad \text{for any } \epsilon > 0.$$

Then Assumption 1.1 is satisfied. Moreover, if additionally

$$\mathbf{P}\{\alpha_\gamma < \epsilon\} \geq c_1 \epsilon^{\delta_1} \quad \text{for sufficiently small } \epsilon > 0$$

for some positive constants c_1 and δ_1 , then Assumption 1.2 is satisfied.

- (ii) **Poisson model.** The random set Γ_ω is the Poisson configuration with intensity measure ρdx , where ρ is some positive constant, i.e. the following holds (see e.g. [30, 2]).

- (a) For any Borel set E with finite Lebesgue measure $|E|$,

$$\mathbf{P}\{\#(\Gamma_\omega \cap E) = j\} = e^{-\rho|E|} \frac{(\rho|E|)^j}{j!} \quad (j = 0, 1, 2, \dots),$$

where $\#S$ denotes the number of the points in the set S .

- (b) For any disjoint Borel sets E_1, \dots, E_n with finite Lebesgue measure, the random variables $\{\#(\Gamma_\omega \cap E_j)\}_{j=1}^n$ are independent.

The fluxes $\{\alpha_\gamma\}$ are i.i.d. random variables independent of Γ_ω and satisfying $\mathbf{E}[\alpha_\gamma] > 0$ (α_γ can be a constant sequence). Then Assumption 1.2 holds, since

$$\mathbf{P}\{\Phi_\omega(Q_0) = 0\} \geq e^{-\rho} > 0.$$

- (iii) **Accumulating lattice.** This is somewhat an artificial example which satisfies (vi)(a) of Assumption 1.1 but does not satisfy (vi)(b). Considering the i.i.d. assumption ((ii) and (iii) of Assumption 1.1), we give the distribution of Γ_ω only in Q_0 and the distribution of $\alpha_\gamma(\omega)$ for $\gamma \in \Gamma_\omega \cap Q_0$. Let Ω_0 be the set of the positive integers with the probability measure $\mathbf{P}\{m\} = 6/(m\pi)^2$ for $m \in \Omega_0$. For $m \in \Omega_0$, we define

$$\Gamma_m = \left\{ \left(\frac{n_1}{2m+1}, \frac{n_2}{2m+1} \right) \mid n_j \in \mathbb{Z}, |n_j| \leq m \ (j = 1, 2) \right\},$$

$$\alpha_\gamma(m) = \frac{1}{(2m+1)^3} \quad (\gamma \in \Gamma_m).$$

Then, we have $\Phi_m(Q_0) = (2m+1)^{-1}$ and

$$\mathbf{P} \{ \Phi_m(Q_0) < \epsilon \} = \frac{6}{\pi^2} \sum_{m > (\epsilon^{-1}-1)/2} \frac{1}{m^2} > c_1 \epsilon$$

for some positive constant c_1 and sufficiently small $\epsilon > 0$. Moreover, since $\sqrt{\alpha_\gamma(m)} = (2m+1)^{-3/2}$ and $\min_{\gamma \neq \gamma'} |\gamma - \gamma'| = (2m+1)^{-1}$, (3) always holds if we take $c = 1/2$. Thus, for small $\epsilon > 0$, (5) holds with $\delta_1 = 1$. But

$$\sum_{\gamma \in \Gamma_m} \sqrt{\alpha_\gamma} = (2m+1)^{1/2} \geq \sqrt{3}$$

for every $m \in \Omega_0$, so the probability of the event (4) is 0 for any $0 < \epsilon < \sqrt{3}$.

An example satisfying (vi)(b) but not satisfying (vi)(a) can be more easily constructed, by considering the two-point fields approaching very fast to each other as the fluxes tend to 0.

According to [11, section 4], we can construct the vector potential \mathbf{a}_ω satisfying (1) as follows. For notational convenience, we identify $x = (x_1, x_2) \in \mathbb{R}^2$ with $z = x_1 + ix_2 \in \mathbb{C}$. If a meromorphic function $\phi_\omega(z)$ has poles only on Γ_ω and the principal part of ϕ_ω at $z = \gamma$ is $\alpha_\gamma/(z-\gamma)$, then the Cauchy-Riemann relation and the distributional equality $\Delta \log |z-\gamma| = 2\pi\delta_\gamma$ imply the vector potential

$$\mathbf{a}_\omega = (\operatorname{Im} \phi_\omega, \operatorname{Re} \phi_\omega) \tag{7}$$

satisfies (1). Such a meromorphic function ϕ_ω always exists by the Mittag-Leffler theorem. Under Assumption 1.1, the function ϕ_ω is explicitly given by

$$\phi_\omega(z) = \frac{\alpha_0(\omega)}{z} + \sum_{\gamma \in \Gamma_\omega \setminus \{0\}} \alpha_\gamma(\omega) \left(\frac{1}{z-\gamma} + \frac{1}{\gamma} + \frac{z}{\gamma^2} \right),$$

where we put $\alpha_0(\omega) = 0$ if $0 \notin \Gamma$. We can prove that the sum in the above formula converges almost surely under Assumption 1.1 (a similar argument is seen in [26, Proposition 4.1]).

We denote the Friedrichs extension of the operator \mathcal{L}_ω with the operator domain $C_0^\infty(\mathbb{R}^2 \setminus \Gamma_\omega)$ by H_ω , where $C_0^\infty(U)$ denotes the set of the compactly supported smooth functions whose support is contained in U . The operator H_ω is a non-negative self-adjoint operator on $L^2(\mathbb{R}^2)$, and the operator

domain $D(H_\omega)$ of H_ω is given by

$$D(H_\omega) = \{u \in L^2(\mathbb{R}^2) \mid \mathcal{L}_\omega u \in L^2(\mathbb{R}^2), \limsup_{x \rightarrow \gamma} |u(x)| < \infty \text{ for any } \gamma \in \Gamma_\omega\}, \quad (8)$$

where the derivative $\mathcal{L}_\omega u$ is interpreted in the sense of the Schwartz distribution on $\mathbb{R}^2 \setminus \Gamma_\omega$.

For the spectrum $\sigma(H_\omega)$ of H_ω , we obtain the following result.

Theorem 1.3. *Under Assumption 1.1, we have $\sigma(H_\omega) = [0, \infty)$ almost surely.*

There are numerous results similar to Theorem 1.3 in the theory of random Schrödinger operators (see e.g. [5, 31, 2, 19, 17]). Especially, Borg [3, Theorem 4.3.1] proves the special case of Theorem 1.3, in the case Γ_ω is a non-random lattice. Nevertheless, the proof of Theorem 1.3 is not trivial from the following reason.

The main strategy to prove the almost sure spectrum $\Sigma = [0, \infty)$ used in the known results is as follows. First, we find a class of the *admissible operators* \mathcal{A} , such that Σ is expressed as the closure of the union of the spectrum of all the operators belonging to \mathcal{A} (see e.g. Kirsch-Martinelli [19, Theorem 3] or Kirsch [17, Page 305, Theorem 2]). Next, we find a sequence of operators H_n in \mathcal{A} such that H_n converges to the free Laplacian $-\Delta$ in the strong resolvent sense. This implies $\Sigma \supset [0, \infty)$ by [29, Theorem VIII.24], and the converse inclusion is trivial if the operators in \mathcal{A} are non-negative.

However, under Assumption 1.1, finding such a sequence H_n is not an easy task, because the operator domain $D(H_\omega)$ depends on the lattice Γ_ω and the flux α_γ due to the singularity of the vector potential \mathbf{a}_ω (the well-known criterion on the strong resolvent convergence [29, Theorem VIII.25] requires the operators H_n have a common operator core). Moreover, the unboundedness of the number of the lattice points in the basic cell Q_0 (such as our example (iii)) makes the problem more difficult.

In order to overcome this difficulty, we directly construct the Weyl sequence for any energy $E \geq 0$, i.e., the sequence $\{u_n\} \subset D(H_\omega)$ such that $\|(H_\omega - E)u_n\| \rightarrow 0$ and $\|u_n\| = 1$ ($\|\cdot\|$ denotes the L^2 -norm), by multiplying the factor

$$\Psi(z) = \prod_{\gamma \in \Gamma \cap Q} |z - \gamma|^{\alpha_\gamma}$$

(Q is some cube) to the eigenfunction $e^{i\sqrt{E}x_1}$ of $-\Delta$. Under our small flux assumption, we can almost surely choose the cube Q so that the magnetic flux on Q is arbitrarily small, and then we can construct the desired sequence. For the detail, see section 2.

Next we shall introduce the integrated density of states (IDS) for the operator H_ω . Let $H_{\omega,N}^k$ be the self-adjoint realization of the operator \mathcal{L}_ω on $L^2(Q_k)$ with the Neumann boundary conditions $(\frac{1}{i}\nabla - \mathbf{a}_\omega)u \cdot \mathbf{n} = 0$ on ∂Q_k (\mathbf{n} is the unit outer normal at the boundary point; notice also that $\partial Q_k \cap \Gamma_\omega = \emptyset$ almost surely by (v) of Assumption 1.1). For $E \in \mathbb{R}$, let $N_{\omega,N}^k(E)$ be the number of the eigenvalues of $H_{\omega,N}^k$ less than or equal to E , and

$$N(E) = \lim_{k \rightarrow \infty} \frac{1}{|Q_k|} N_{\omega,N}^k(E),$$

where $|\cdot|$ denotes the Lebesgue measure. We can prove the limit $N(E)$ exists almost surely and independent of ω , by Akcoglu-Krengel's superadditive ergodic theorem (see [5, 1]).

Our second result is the following inequality, known as the *Lifshitz tail* estimate in the theory of the random Schrödinger operators.

Theorem 1.4. *(i) Suppose Assumption 1.1 holds. Then, there exist positive constants C and E_0 independent of ω and E , such that*

$$N(E) \leq e^{-\frac{C}{E}} \quad (9)$$

for any E with $0 < E < E_0$.

(ii) Suppose Assumption 1.2 holds. Then we have

$$\lim_{E \rightarrow +0} \frac{\log |\log N(E)|}{\log E} = -1. \quad (10)$$

Notice that the upper bound in (10) is a consequence from (9). Thus (10) gives a lower bound of $N(E)$ in some weak sense.

The Lifshitz tail estimate is first predicted by I. M. Lifshitz [23, 24], and has been studied in connection with the mathematical proof of the Anderson localization, mainly for the Schrödinger operators with random *scalar potentials*. For the reference, see e.g. [5, 31, 18, 10, 32, 14].

The mathematical proof of the Lifshitz tail and the Anderson localization for the Schrödinger operators with random *magnetic fields* is comparatively

difficult, mainly because of the following two reasons. First, the eigenvalues of the operator restricted to a finite box do *not* depend monotonically on the random coupling constants. This fact makes the proof of the Wegner estimate rather difficult. Second, especially in the two-dimensional case, the correlation between the values of the magnetic vector potential at two different points is rather strong, since any vector potential corresponding to a single-site magnetic field falls off at infinity not faster than $O(|x|^{-1})$, if the total magnetic flux is not zero.

The second difficulty can be solved if we assume the single-site magnetic vector potential is compactly supported. Under this assumption, Klopp–Nakamura–Nakano–Nomura [20] prove the Anderson localization in the discrete model. In the continuum model, Ghribi [12] proves the internal Lifshitz tail, and Ghribi–Hislop–Klopp [13] prove the Anderson localization.

In the case the dimension is two and the magnetic flux of the single-site magnetic field is non-zero, there is only a few results. Nakamura [27, 28] proves the Lifshitz tail at the bottom of the spectrum, both in the discrete and in the continuum model. Erdős–Hasler [7, 8, 9] give remarkable results, in which they succeed to prove the Anderson localization both in the discrete and in the continuum model (though the form of the magnetic field is rather special in the continuum model). Ueki [33, 34] studies the Gaussian random magnetic fields, and obtain the Wegner estimate in [35] by using the idea of Erdős–Hasler. Hasler–Luckett [15] also prove the Wegner estimate with optimal volume dependence in the discrete model.

There are also some results for the random δ magnetic fields. Borg–Pulé [4] prove the Lifshitz tail for a smooth approximation of a random δ magnetic field, but not for the δ magnetic fields itself. Borg [3] gives a stochastic representation of the Laplace transform of IDS for the Schrödinger operator with δ magnetic fields, in terms of the rotation number of the two-dimensional Brownian motion. However, there seems to be no results for the Lifshitz tail for random δ magnetic fields at present.

The strategy for the proof of Theorem 1.4 is as follows. In Nakamura’s paper [27], the crucial inequality in the proof of the Lifshitz tail is the Avron–Herbst–Simon estimate:

$$H_\omega \geq \text{curl } \mathbf{a}_\omega. \quad (11)$$

If the magnetic field is regular, we can reduce the problem to the scalar potential case by using (11). However, in our case the inequality (11) is no longer useful, since $\text{curl } \mathbf{a}_\omega = 0$ almost everywhere. Instead of (11), we use

the Hardy-type inequality by Laptev–Weidl [21] (see also (46) below). Then we can reduce the problem to the scalar potential case as in [27].

For the proof of the lower bound, we follow the standard strategy given in [5, Theorem VI.2.7]. We give an estimate for the probability of the event ‘the lowest eigenvalue of the Dirichlet realization $H_{\omega,D}^k$ of the operator H_ω on Q_k is less than ϵ ’, by constructing an approximation of the ground state explicitly. Here we use the estimates obtained in the proof of Theorem 1.3.

The rest of the paper is organized as follows. In section 2, we shall prove Theorem 1.3, and the lower bound in Theorem 1.4. In section 3, we shall introduce the Hardy-type inequality by Laptev–Weidl [21], and give some key inequality in the proof of Theorem 1.4. In section 4, we shall prove Theorem 1.4.

2 Spectrum

In this section, we shall give a proof of Theorem 1.3, using the method of approximating eigenfunctions. First we review a lemma about the gauge transform for δ magnetic fields.

Lemma 2.1. *Let U be a simply connected open set in \mathbb{R}^2 and Γ be a discrete subset of U without accumulation points in U . Let $\mathbf{a}, \tilde{\mathbf{a}} \in C^\infty(U \setminus \Gamma; \mathbb{R}^2) \cap L_{\text{loc}}^1(U; \mathbb{R}^2)$. Assume*

$$\text{curl } \mathbf{a} = \text{curl } \tilde{\mathbf{a}} + \sum_{\gamma \in \Gamma} 2\pi n_\gamma \delta_\gamma$$

holds in $\mathcal{D}'(U)$, where $n_\gamma \in \mathbb{Z}$ and δ_γ is the Dirac measure supported on the point γ . Then, there exists $\Phi \in C^\infty(U \setminus \Gamma)$ such that $|\Phi(z)| = 1$ for any $z \in U \setminus \Gamma$ and

$$\left(\frac{1}{i} \nabla - \mathbf{a} \right) (\Phi u) = \Phi \left(\frac{1}{i} \nabla - \tilde{\mathbf{a}} \right) u$$

for any $u \in C^\infty(U \setminus \Gamma)$.

For a proof, see [11, Theorem 3.1]. By Lemma 2.1, we can arbitrarily choose an appropriate vector potential for given δ -magnetic fields.

Proof of Theorem 1.3. Since the operator H_ω is non-negative, we have $\sigma(H_\omega) \subset [0, \infty)$. In order to prove $\sigma(H_\omega) \supset [0, \infty)$, we shall prove $\xi^2 \in \sigma(H_\omega)$ almost

surely for any $\xi \in \mathbb{Q}$. Then it suffices to show that we can almost surely find a function u satisfying

$$u \in D(H_\omega), \quad \|u\| = 1, \quad \|(H_\omega - \xi^2)u\| < \epsilon \quad (12)$$

for any positive rational number ϵ , where the norm without suffix denotes the L^2 -norm.

Take sufficiently large positive integers k and l , which will be determined later. By (vi) of Assumptions 1.1, we can almost surely find $n \in \mathbb{Z}^2$ such that the square $Q = Q_k + n$ satisfies

$$\Phi_\omega(m + Q_0) < \frac{1}{l} \quad \text{for any } m \in Q \cap \mathbb{Z}^2, \quad (13)$$

$$B_\gamma(c\sqrt{\alpha_\gamma}) \cap B_{\gamma'}(c\sqrt{\alpha_{\gamma'}}) = \emptyset, \quad B_\gamma(c\sqrt{\alpha_\gamma}) \cap \partial(m + Q_0) = \emptyset \\ \text{for any } m \in Q \cap \mathbb{Z}^2, \quad \gamma, \gamma' \in (m + Q_0) \cap \Gamma_\omega \text{ with } \gamma \neq \gamma', \quad (14)$$

or

$$\Phi_\omega(m + Q_0) \leq \sum_{\gamma \in (m + Q_0) \cap \Gamma_\omega} \sqrt{\alpha_\gamma} < \frac{1}{l} \quad \text{for any } m \in Q \cap \mathbb{Z}^2. \quad (15)$$

We omit the random parameter ω in the rest of the proof, because we do not use the probabilistic argument hereafter.

We shall construct the function u supported in the square Q . By Lemma 2.1 and (7), we may assume

$$\mathbf{a}(z) = (\operatorname{Im} \psi(z), \operatorname{Re} \psi(z)), \quad \psi(z) = \sum_{\gamma \in \Gamma \cap Q} \frac{\alpha_\gamma}{z - \gamma}$$

in Q , where we again identify $x = (x_1, x_2)$ with $z = x_1 + ix_2$ and regard γ as a complex number. Put

$$\Psi(z) = \prod_{\gamma \in \Gamma \cap Q} |z - \gamma|^{\alpha_\gamma}.$$

If $\Gamma \cap Q = \emptyset$, we put $\psi(z) = 0$ and $\Psi(z) = 1$. Then we have

$$2\partial_{\bar{z}}\Psi(z) = \overline{\psi(z)}\Psi(z), \quad -2\partial_z\Psi(z)^{-1} = \psi(z)\Psi(z)^{-1}, \quad (16)$$

where $\partial_z = (\partial_1 - i\partial_2)/2$, $\partial_{\bar{z}} = (\partial_1 + i\partial_2)/2$. Thus we have by (16)

$$\begin{aligned} \mathcal{L} &= \left(\frac{1}{i} \nabla - \mathbf{a} \right)^2 \\ &= (2\partial_{\bar{z}} + \overline{\psi}) (-2\partial_z + \psi) \\ &= \Psi^{-1} (2\partial_{\bar{z}}) \Psi^2 (-2\partial_z) \Psi^{-1}, \end{aligned} \quad (17)$$

as an operator acting on the functions on $Q \setminus \Gamma$.

Let $\chi_k \in C_0^\infty(Q)$ satisfying the following conditions:

$$0 \leq \chi_k(z) \leq 1, \quad \chi_k(z) = \begin{cases} 1 & (z \in n + Q_{k-1}), \\ 0 & (z \in n + (Q_k \setminus Q_{k-1/2})), \end{cases}$$

$$\|\nabla \chi_k\|_\infty + \|\Delta \chi_k\|_\infty \leq C_0,$$

where C_0 is a constant independent of k, n . Put

$$v_k = \chi_k \Psi e^{ix_1 \xi}, \quad u_k = \frac{v_k}{\|v_k\|}.$$

We are going to show $u = u_k$ satisfies (12), if we take k and l sufficiently large.

Let us estimate $\|v_k\|$ from below. Take $m \in (n + Q_{k-1}) \cap \mathbb{Z}^2$, and put

$$\begin{aligned} \Gamma_1 &= \Gamma \cap (m + Q_1), \\ \Gamma_2 &= \Gamma \cap (Q \setminus (m + Q_1)). \end{aligned}$$

Let $z \in m + Q_0$. Using the inequality $e^t \geq 1 + t$ ($t \in \mathbb{R}$), we have

$$\begin{aligned} \prod_{\gamma \in \Gamma_1} |z - \gamma|^{2\alpha_\gamma} &= \exp \left(\sum_{\gamma \in \Gamma_1} 2\alpha_\gamma \log |z - \gamma| \right) \\ &\geq 1 + \sum_{\gamma \in \Gamma_1} 2\alpha_\gamma \log |z - \gamma|, \end{aligned} \tag{18}$$

$$\prod_{\gamma \in \Gamma_2} |z - \gamma|^{2\alpha_\gamma} \geq 1. \tag{19}$$

Notice that

$$\int_S \log |z| dx \geq \int_{|z| \leq 1} \log |z| dx = 2\pi \int_0^1 r \log r dr = -\frac{\pi}{2} \tag{20}$$

for any bounded Borel set S . By (18), (19), (20), and (13) or (15), we have

$$\begin{aligned} \int_{m+Q_0} |v_k|^2 dx &\geq 1 + \sum_{\gamma \in \Gamma_1} \int_{m+Q_0} 2\alpha_\gamma \log |z - \gamma| dx \\ &\geq 1 - \pi \sum_{\gamma \in \Gamma_1} \alpha_\gamma \\ &\geq 1 - \frac{9\pi}{l} \end{aligned} \tag{21}$$

for any $m \in n + Q_{k-1}$. Take l_0 so large that $1 - 9\pi/l_0 > 0$. Then, summing up the both sides of (21) with respect to m , we find a positive constant C_1 independent of k, l such that

$$\|v_k\| \geq C_1(2k+1) \quad (22)$$

for any $l \geq l_0$.

Next we shall estimate $\|(H - \xi^2)v_k\|$ from above. Since

$$2\partial_z e^{ix_1\xi} = 2\partial_{\bar{z}} e^{ix_1\xi} = \partial_1 e^{ix_1\xi} = i\xi e^{ix_1\xi},$$

we have by (16) and (17)

$$\begin{aligned} \mathcal{L}v_k &= \Psi^{-1}(2\partial_{\bar{z}})\Psi^2(-2\partial_z)(\chi_k e^{ix_1\xi}) \\ &= \Psi^{-1}(2\partial_{\bar{z}})\Psi^2(-2\partial_z\chi_k - i\xi\chi_k) e^{ix_1\xi} \\ &= -2\Psi\bar{\psi}(2\partial_z\chi_k + i\xi\chi_k) e^{ix_1\xi} \\ &\quad -\Psi(\Delta\chi_k + 2i\xi\partial_1\chi_k) e^{ix_1\xi} + \xi^2 v_k. \end{aligned} \quad (23)$$

(23) implies the singularity of $\mathcal{L}v_k$ near $\gamma \in Q \cap \Gamma$ is at most $O(|z - \gamma|^{-1+\alpha_\gamma})$, so $\mathcal{L}v_k \in L^2(\mathbb{R}^2)$ and $v_k \in D(H)$ by (8). By (23), we have

$$\|(H - \xi^2)v_k\| \leq C_2 (\|\Psi\psi\|_{L^2(Q)} + \|\Psi\|_{L^2(n+(Q_k \setminus Q_{k-1}))}), \quad (24)$$

where $C_2 = 2(C_0 + 1)(|\xi| + 1)$.

In the sequel, we denote $d_k = \text{diam } Q = \sqrt{2}(2k+1)$. Then we have

$$\|\Psi\|_{L^2(n+(Q_k \setminus Q_{k-1}))} \leq \|\Psi\|_\infty |Q_k \setminus Q_{k-1}|^{\frac{1}{2}} \leq d_k^{\Phi(Q)} \sqrt{8k}. \quad (25)$$

We shall estimate $\|\Psi\psi\|_{L^2(Q)}$ under the conditions (13) and (14), and under the condition (15), separately. In both cases, we have

$$\Phi(Q) < \frac{(2k+1)^2}{l}, \quad (26)$$

and

$$|\Psi\psi(z)| \leq \sum_{\gamma \in \Gamma \cap Q} \alpha_\gamma |z - \gamma|^{\alpha_\gamma - 1} \prod_{\gamma' \neq \gamma} |z - \gamma'|^{\alpha_{\gamma'}} \quad (27)$$

for $z \in Q$, where $\prod_{\gamma' \neq \gamma}$ stands for $\prod_{\gamma' \in \Gamma \cap Q, \gamma' \neq \gamma}$ as in the sequel.

Assume (13) and (14) hold. By (27), we have

$$\begin{aligned}
& \|\Psi\psi\|_{L^2(\bigcup_{\gamma} B_{\gamma}(c\sqrt{\alpha_{\gamma}}))}^2 \\
& \leq \sum_{\gamma \in \Gamma \cap Q} 2\alpha_{\gamma}^2 \int_{B_{\gamma}(c\sqrt{\alpha_{\gamma}})} |z - \gamma|^{2\alpha_{\gamma}-2} dx \cdot d_k^{2\Phi(Q)} \\
& \quad + \sum_{\gamma \in \Gamma \cap Q} 2 \int_{B_{\gamma}(c\sqrt{\alpha_{\gamma}})} \left(\sum_{\mu \neq \gamma} \alpha_{\mu} |z - \mu|^{\alpha_{\mu}-1} \right)^2 dx \cdot d_k^{2\Phi(Q)}. \quad (28)
\end{aligned}$$

The first sum in (28) is bounded by

$$\begin{aligned}
& \sum_{\gamma \in \Gamma \cap Q} 2\alpha_{\gamma}^2 \cdot 2\pi \int_0^{c\sqrt{\alpha_{\gamma}}} r^{2\alpha_{\gamma}-2} \cdot r dr \cdot d_k^{2\Phi(Q)} \\
& = \sum_{\gamma \in \Gamma \cap Q} 2\pi \alpha_{\gamma} (c\sqrt{\alpha_{\gamma}})^{2\alpha_{\gamma}} \cdot d_k^{2\Phi(Q)} \\
& \leq 2\pi \Phi(Q) d_k^{2\Phi(Q)} \quad (29)
\end{aligned}$$

where we used $0 < c \leq 1$ and $0 \leq \alpha_{\gamma} < 1$ in the last inequality.

For the second sum in (28), we use the inequality

$$\frac{1}{2} \leq \frac{|z - w|}{|z - \mu|} \leq \frac{3}{2}$$

for $z \notin B_{\mu}(c\sqrt{\alpha_{\mu}})$ and $w \in B_{\mu}(c\sqrt{\alpha_{\mu}}/2)$. Then we see that there exists a positive constant C_3 independent of μ and α_{μ} with $0 \leq \alpha_{\mu} < 1$ such that

$$\alpha_{\mu} |z - \mu|^{\alpha_{\mu}-1} \leq C_3 \int_{B_{\mu}(c\sqrt{\alpha_{\mu}}/2)} |z - w|^{\alpha_{\mu}-1} du \quad (30)$$

for $z \notin B_{\mu}(c\sqrt{\alpha_{\mu}})$, where we write $w = u_1 + iu_2$ and $du = du_1 du_2$. Summing up (30) with respect to $\mu \neq \gamma$, we have

$$\sum_{\mu \neq \gamma} \alpha_{\mu} |z - \mu|^{\alpha_{\mu}-1} \leq C_3 \int_{\bigcup_{\mu \neq \gamma} B_{\mu}(c\sqrt{\alpha_{\mu}}/2)} (\min(|z - w|, 1))^{-1} du \quad (31)$$

for $z \in B_{\gamma}(c\sqrt{\alpha_{\gamma}})$, where we used the disjointness assumption (14). The area of the domain of integration in (31) is bounded by

$$\frac{\pi c^2}{4} \sum_{\mu \neq \gamma} \alpha_{\mu} \leq \frac{\pi c^2}{4} \Phi(Q),$$

and the right-hand side equals the area of the disc of radius $c\sqrt{\Phi(Q)}/2$. Since the integrand of (31) is monotone non-increasing with respect to $|z - w|$, we have by (31)

$$\begin{aligned} \sum_{\mu \neq \gamma} \alpha_\mu |z - \mu|^{\alpha_\mu - 1} &\leq C_3 \int_{|z - w| \leq c\sqrt{\Phi(Q)}/2} |z - w|^{-1} du \\ &= \pi c C_3 \sqrt{\Phi(Q)}, \end{aligned} \quad (32)$$

provided that l is sufficiently large so that

$$\frac{c\sqrt{\Phi(Q)}}{2} \leq \frac{c(2k+1)}{2\sqrt{l}} \leq 1, \quad (33)$$

where we used (26). By (32), the second sum in (28) is bounded by

$$\sum_{\gamma \in \Gamma \cap Q} 2 \cdot \pi c^2 \alpha_\gamma \cdot (\pi c C_3)^2 \Phi(Q) \cdot d_k^{2\Phi(Q)} = C_4 \Phi(Q)^2 d_k^{2\Phi(Q)} \quad (34)$$

for sufficiently large l satisfying (33), where $C_4 = 2\pi^3 c^4 C_3^2$.

For $z \in Q \setminus \bigcup_\gamma B_\gamma(c\sqrt{\alpha_\gamma})$, we have estimate (30) for every $\mu \in Q \cap \Gamma$. Repeating the above argument, we see that

$$|\Psi\psi(z)| \leq \pi c C_3 \sqrt{\Phi(Q)} \cdot d_k^{\Phi(Q)}$$

for sufficiently large l satisfying (33), and

$$\|\Psi\psi\|_{L^2(Q \setminus \bigcup_\gamma B_\gamma(c\sqrt{\alpha_\gamma}))}^2 \leq (\pi c C_3)^2 \Phi(Q) d_k^{2\Phi(Q)} \cdot (2k+1)^2, \quad (35)$$

since the area of Q is $(2k+1)^2$. By (26), (28), (29), (34) and (35), there exists a positive constant C_5 independent of k and l such that

$$\|\Psi\psi\|_{L^2(Q)} \leq C_5 \frac{(2k+1)^2}{\sqrt{l}} d_k^{\Phi(Q)}, \quad (36)$$

for sufficiently large l satisfying (33).

By (22), (24), (25) and (36), we have

$$\|(H - \xi^2)u_k\| \leq C_1^{-1} C_2 d_k^{(2k+1)^2/l} \left(\frac{C_5(2k+1)}{\sqrt{l}} + \frac{\sqrt{8k}}{2k+1} \right) \quad (37)$$

for sufficiently large l satisfying $l \geq l_0$ and (33). For $\epsilon > 0$, take k so large that $C_1^{-1}C_2\sqrt{8k}/(2k+1) < \epsilon/2$, and then take l so large that the right hand side of (37) is less than ϵ . Then u_k satisfies (12).

Next we assume (15) holds. By the Minkowski inequality, we have

$$\begin{aligned}
& \|\Psi\psi\|_{L^2(Q)} \\
& \leq \sum_{\gamma \in Q \cap \Gamma} \alpha_\gamma \left(\int_Q |z - \gamma|^{2\alpha_\gamma - 2} \prod_{\gamma' \neq \gamma} |z - \gamma'|^{2\alpha_{\gamma'}} dx \right)^{\frac{1}{2}} \\
& \leq \sum_{\gamma \in Q \cap \Gamma} \alpha_\gamma d_k^{\Phi(Q) - \alpha_\gamma} \left(2\pi \int_0^{d_k} r^{2\alpha_\gamma - 2} \cdot r dr \right)^{\frac{1}{2}} \\
& = \sum_{\gamma \in Q \cap \Gamma} \sqrt{\alpha_\gamma \pi} d_k^{\Phi(Q)} \\
& \leq \frac{\sqrt{\pi}}{l} (2k+1)^2 d_k^{\Phi(Q)}. \tag{38}
\end{aligned}$$

By (22), (24), (25), (26) and (38), we have

$$\|(H - \xi^2)u_k\| \leq C_1^{-1}C_2 d_k^{(2k+1)^2/l} \left(\frac{\sqrt{\pi}(2k+1)}{l} + \frac{\sqrt{8k}}{2k+1} \right)$$

for sufficiently large l , and obtain the conclusion similarly. \square

Using the estimates obtained in the above proof, we can also prove the lower bound in (ii) of Theorem 1.4. We use the same notation introduced above.

Proof of the lower bound in (ii) of Theorem 1.4. Let $H_{\omega,D}^k$ be the operator H_ω restricted on $L^2(Q_k)$ with the Dirichlet boundary conditions, and put $N_{\omega,D}^k(E)$ be the number of the eigenvalues of $H_{\omega,D}^k$ less than or equal to E , counted with multiplicity. It is known that

$$N(E) = \sup_{k \geq 1} \frac{1}{|Q_k|} N_{\omega,D}^k(E)$$

holds almost surely (see [5, VI.1.3]). Let $E_1(H)$ denotes the smallest eigenvalue of a self-adjoint operator H . Then we have

$$\begin{aligned}
N(E) & \geq \frac{1}{|Q_k|} \mathbf{E} [N_{\omega,D}^k(E)] \\
& \geq \frac{1}{|Q_k|} \mathbf{P} [E_1(H_{\omega,D}^k) \leq E] \tag{39}
\end{aligned}$$

for any $k \geq 1$.

We assume (5) holds for sufficiently small $\epsilon > 0$, and estimate the right hand side of (39) from below. Let ϵ be a small positive number. Take positive integers k and l so that

$$\frac{\epsilon}{8} < \left(\frac{\pi}{2k+1} \right)^2 < \frac{\epsilon}{4}, \quad \frac{\epsilon^3}{2} < \frac{1}{l} < \epsilon^3. \quad (40)$$

Then (33) is satisfied for sufficiently small ϵ .

Let us suppose (13) and (14) hold with $Q = Q_k$, and construct an approximation of the ground state of $H_{\omega,D}^k$. Let $-\Delta_D^k$ be the Dirichlet Laplacian on Q_k . The ground state of $-\Delta_D^k$ is

$$f_k(x) = \cos \left(\frac{\pi}{2k+1} x_1 \right) \cos \left(\frac{\pi}{2k+1} x_2 \right),$$

with the ground state energy $2 \cdot (\pi/(2k+1))^2$. Put

$$w_k = \Psi f_k, \quad u_k = \frac{w_k}{\|w_k\|}.$$

Then, similar to (22), we can prove

$$\|w_k\| \geq C'_1(2k+1) \quad (41)$$

for any $l \geq l_0$ and $k \geq 1$, by using the fact $|f_k| \geq 1/4$ on $Q_{k/2}$. By (16) and (17), we have

$$H_{\omega,D}^k w_k = 2 \left(\frac{\pi}{2k+1} \right)^2 w_k + 2\Psi \bar{\psi} \cdot (-2\partial_z f_k).$$

Since $|2\partial_z f_k| \leq \pi/(2k+1)$, we have by (41)

$$(u_k, H_{\omega,D}^k u_k) \leq 2 \left(\frac{\pi}{2k+1} \right)^2 + \frac{2\pi C_1'^{-1}}{(2k+1)^2} \|\Psi \psi\|, \quad (42)$$

where $(u, v) = \int \bar{u} v dx$ denotes the L^2 -inner product. Then (42), (26), (36) and (40) imply

$$d_k = \sqrt{2}(2k+1) < 4\pi\epsilon^{-1/2}, \quad \Phi(Q) < \frac{(2k+1)^2}{l} < 8\pi^2\epsilon^2,$$

$$\begin{aligned}
(u_k, H_{\omega,D}^k u_k) &\leq 2 \left(\frac{\pi}{2k+1} \right)^2 + \frac{2\pi C_1'^{-1} C_5}{\sqrt{l}} d_k^{\Phi(Q)} \\
&\leq \epsilon \left(\frac{1}{2} + 2\pi C_1'^{-1} C_5 \epsilon^{1/2} (4\pi \epsilon^{-1/2})^{8\pi^2 \epsilon^2} \right).
\end{aligned}$$

Since the expression in the big parenthesis tends to $1/2$ as ϵ tends to 0, we have by the min-max principle

$$E_1(H_{\omega,D}^k) < \epsilon \quad (43)$$

for sufficiently small ϵ . Thus the events (13) and (14) with k and l satisfying (40) imply the inequality (43). Then the independentness assumption implies

$$\begin{aligned}
\mathbf{P}\{E_1(H_{\omega,D}^k \leq \epsilon)\} &\geq \mathbf{P}\{(13) \text{ and } (14) \text{ holds}\} \\
&\geq (c_1 l^{-\delta_1})^{|Q_k|} \\
&\geq (c_1 (\epsilon^3/2)^{\delta_1})^{8\pi^2 \epsilon^{-1}}
\end{aligned} \quad (44)$$

for sufficiently small $\epsilon > 0$. Then (39), (40) and (44) imply

$$\liminf_{E \rightarrow +0} \frac{\log |\log N(E)|}{\log E} \geq -1,$$

which is the desired conclusion. We can give the same conclusion in the case (6) holds for sufficiently small $\epsilon > 0$, by using (38) instead of (36). \square

3 Hardy-type inequality

For $d \geq 3$, the Hardy inequality says

$$\int_{\mathbb{R}^d} |\nabla u|^2 dx \geq \left(\frac{d-2}{2} \right)^2 \int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2} dx \quad (45)$$

for any $u \in H^1(\mathbb{R}^d)$. The inequality (45) fails when $d = 2$, however, Laptev-Weidl [21] prove that a similar inequality holds if there exists a δ magnetic field at the origin.

Lemma 3.1 (Laptev-Weidl). *Let $\alpha \in \mathbb{R}$ and put $\mathbf{a}_\alpha = \left(\operatorname{Im} \frac{\alpha}{z}, \operatorname{Re} \frac{\alpha}{z} \right)$, where $z = x_1 + ix_2$ (a_α satisfies $\operatorname{curl} \mathbf{a}_\alpha = 2\pi\alpha\delta$). Then, we have*

$$\int_{|x| \leq R} \left| \left(\frac{1}{i} \nabla - \mathbf{a}_\alpha \right) u \right|^2 dx \geq \rho(\alpha) \int_{|x| \leq R} \frac{|u|^2}{|x|^2} dx \quad (46)$$

for any $R > 0$ and any $u \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$, where $\rho(\alpha) = \min_{n \in \mathbb{Z}} |n - \alpha|^2$.

Let us return to our model. We define a random scalar potential $V_\omega(z)$ as follows. For $m \in \mathbb{Z}^2$, put $\Gamma_{\omega,m} = (m + Q_0) \cap \Gamma_\omega$ and

$$\delta_m = \min_{\gamma \in \Gamma_{\omega,m}} (\text{dist}(\gamma, (m + \partial Q_0) \cup (\Gamma_{\omega,m} \setminus \{\gamma\}))) / 2.$$

For $z \in m + Q_0$, put

$$V_\omega(z) = \begin{cases} \min \left(\frac{1}{\delta_m^2} \rho(\alpha_\gamma(\omega)), 1 \right) & (\text{if } z \in B_\gamma(\delta_m) \text{ for some } \gamma \in \Gamma_{\omega,m}), \\ 0 & (\text{otherwise}). \end{cases}$$

Lemma 3.2. *Let $H_{\omega,N}^k$ be the Neumann realization of \mathcal{L}_ω on Q_k , and Δ_N^k the Neumann Laplacian on Q_k . Then,*

$$E_1(H_{\omega,N}^k) \geq E_1 \left(\frac{1}{2} (-\Delta_N^k + V_\omega) \right).$$

Proof. By using Lemma 2.1 and Lemma 3.1, we have

$$\int_{m+Q_0} \left| \left(\frac{1}{i} \nabla - \mathbf{a}_\omega \right) u(z) \right|^2 dx \geq \int_{m+Q_0} V_\omega(z) |u(z)|^2 dx dy \quad (47)$$

for any $u \in C_0^\infty(\mathbb{R}^2 \setminus \Gamma_\omega)$ and any $m \in \mathbb{Z}^2$. Notice that³

$$\nabla |u| = \text{Re}(\text{sgn } \bar{u} \nabla u) = \text{Re}(\text{sgn } \bar{u} (\nabla - i \mathbf{a}_\omega) u) \quad \text{a.e.} \quad (48)$$

holds for $u \in C_0^\infty(\mathbb{R}^2 \setminus \Gamma_\omega)$, where $\text{sgn } z = z/|z|$ for $z \neq 0$ and $\text{sgn } 0 = 0$. Taking the absolute value of the both sides, we have

$$\left| \left(\frac{1}{i} \nabla - \mathbf{a}_\omega \right) u \right|^2 \geq |\nabla |u||^2 \quad \text{a.e.} \quad (49)$$

By (47) and (49), we have

$$\int_{Q_k} \left| \left(\frac{1}{i} \nabla - \mathbf{a}_\omega \right) u \right|^2 dx \geq \frac{1}{2} \int_{Q_k} \left(|\nabla |u||^2 + V_\omega |u|^2 \right) dx \quad (50)$$

for any $u \in C_0^\infty(\mathbb{R}^2 \setminus \Gamma_\omega)$. Then the conclusion follows immediately from (50) and the min-max principle. \square

³ The equality (48) makes sense for $u \in H_{\text{loc}}^{1,1}(\mathbb{R}^2)$; see e.g. [22, appendix].

4 Proof of Theorem 1.4

By virtue of Lemma 3.2, we can prove Theorem 1.4 as in the scalar potential case (this idea is taken from Nakamura's paper [27]). The proof below is based on Stollmann's book [31].

Lemma 4.1. *There exists a positive constant C_6 independent of Γ_ω , $\alpha(\omega)$, E , and k such that*

$$N_{\omega,N}^k(E) \leq C_6 |Q_k|$$

for any $E \leq 1$ and any non-negative integer k .

Proof. We use the diamagnetic inequality

$$|(H_{\omega,N}^k + \lambda)^{-1}u|(x) \leq (-\Delta_N^k + \lambda)^{-1}|u|(x) \quad \text{a.e.} \quad (51)$$

for any $\lambda > 0$ and $u \in L^2(Q_k)$, where Δ_N^k is the Neumann Laplacian on Q_k ⁴.

Taking u as an approximation of the Dirac measure in (51), we have

$$|(H_{\omega,N}^k + \lambda)^{-1}(x, y)| \leq (-\Delta_N^k + \lambda)^{-1}(x, y) \quad \text{a.e.,}$$

where $T(x, y)$ denotes the integral kernel of an integral operator T . This implies

$$\|(H_{\omega,N}^k + \lambda)^{-1}\|_{\mathcal{I}_2} \leq \|(-\Delta_N^k + \lambda)^{-1}\|_{\mathcal{I}_2}, \quad (52)$$

where $\|\cdot\|_{\mathcal{I}_2}$ denotes the Hilbert-Schmidt norm. Let $l = 2k + 1$ be the length of the edge of Q_k . By (52), we have for any $E \leq 1$

$$\begin{aligned} N_{\omega,N}^k(E) &\leq \text{tr } \chi_{(-\infty, 1]}(H_{\omega,N}^k) \\ &\leq \text{tr } 4(H_{\omega,N}^k + 1)^{-2} \\ &= 4\|(H_{\omega,N}^k + 1)^{-1}\|_{\mathcal{I}_2}^2 \\ &\leq 4\|(-\Delta_N^k + 1)^{-1}\|_{\mathcal{I}_2}^2 \\ &= 4|Q_k| \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} F\left(\frac{m}{l}, \frac{n}{l}\right) \frac{1}{l^2}, \end{aligned}$$

⁴ The diamagnetic inequality on the whole plane is proved for $\mathbf{a} \in L_{\text{loc}}^2(\mathbb{R}^2; \mathbb{R}^2)$ by Leinfelder–Simader [22, Lemma 6], and for the Aharonov–Bohm field by Melgaard–Ouhabaz–Rozenblum [25, Theorem 1.1]. The diamagnetic inequality (51) for the Schrödinger operators with the Neumann boundary conditions can be proved similarly, by slightly changing the proof as the functions in the proof belong to the appropriate form/operator domain (a similar argument is seen in Doi–Iwatsuka–Mine [6, Proposition 3.2]).

where $F(x, y) = 1/(\pi^2(x^2 + y^2) + 1)^2$. The double sum in the last expression converges to

$$\int_0^\infty \int_0^\infty F(x, y) dx < \infty$$

as $k \rightarrow \infty$, so it is bounded with respect to k . Thus we have the conclusion. \square

It is known that

$$N(E) = \inf_{k \geq 1} \frac{1}{|Q_k|} \mathbf{E} [N_{\omega, N}^k(E)]$$

holds almost surely (see [5, VI.1.3]). Let $E_1(H_{\omega, N}^k)$ be the smallest eigenvalue of $H_{\omega, N}^k$, and $\chi(\omega)$ the characteristic function of the event ' $E_1(H_{\omega, N}^k) \leq E$ '. Then we have for every $k \geq 1$ and $E \leq 1$

$$\begin{aligned} N(E) &\leq \frac{1}{|Q_k|} \mathbf{E}[N_{\omega, N}^k(E)] \\ &= \frac{1}{|Q_k|} \mathbf{E}[N_{\omega, N}^k(E) \chi(\omega)] \\ &\leq C_6 \mathbf{P}\{E_1(H_{\omega, N}^k) \leq E\} \\ &\leq C_6 \mathbf{P}\left\{E_1\left(\frac{1}{2}(-\Delta_N^k + V_\omega)\right) \leq E\right\}, \end{aligned} \quad (53)$$

where we used Lemma 4.1 in the second inequality, and Lemma 3.2 in the last inequality.

For $t \in [-1, 1]$, let $E_1(\omega, t)$ be the lowest eigenvalue of $(-\Delta_N^k + tV_\omega)/2$, and $\phi(\omega, t)$ the normalized eigenfunction corresponding to $E_1(\omega, t)$. In particular, $E_1(\omega, 0) = 0$ and $\phi(\omega, 0) = 1/\sqrt{|Q_k|}$. Since $E_1(\omega, 0)$ is a simple eigenvalue, we can assume $E_1(\omega, t)$ and $\phi(\omega, t)$ are differentiable at $t = 0$ by the analytic perturbation theory [16]. By the Feynman-Hellmann theorem [31, Theorem 4.1.29], we have

$$E_1'(\omega, 0) = \frac{1}{2} (V_\omega \phi(\omega, 0), \phi(\omega, 0)) = \frac{1}{2|Q_k|} \int_{Q_k} V_\omega(z) dx, \quad (54)$$

where $'$ denotes the derivative with respect to t . For $n \in \mathbb{Z}^2$, put

$$\beta_n(\omega) = \frac{1}{2} \int_{n+Q_0} V_\omega(z) dx.$$

Then, the random variables $\{\beta_n\}_{n \in \mathbb{Z}^2}$ are i.i.d. and we have by (54)

$$E'_1(\omega, 0) = \frac{1}{|Q_k|} \sum_{n \in Q_k \cap \mathbb{Z}^2} \beta_n(\omega). \quad (55)$$

Lemma 4.2. *Let*

$$s_0 = -\frac{1}{2} \log \mathbf{E}[\exp(-\beta_0(\omega))].$$

Then, we have

$$\mathbf{P} \left\{ \frac{1}{|Q_k|} \sum_{n \in Q_k \cap \mathbb{Z}^2} \beta_n(\omega) \leq s_0 \right\} \leq e^{-s_0 |Q_k|}. \quad (56)$$

Proof. By (iv) of Assumption 1.1, the random variable $\beta_n(\omega)$ takes positive value with positive probability. Thus s_0 is positive and

$$\begin{aligned} (\text{l.h.s. of (56)}) &= \mathbf{P} \left\{ \sum_{n \in Q_k \cap \mathbb{Z}^2} \beta_n(\omega) \leq s_0 |Q_k| \right\} \\ &\leq \mathbf{E} \left[\exp \left(s_0 |Q_k| - \sum_{n \in Q_k \cap \mathbb{Z}^2} \beta_n(\omega) \right) \right] \\ &= e^{s_0 |Q_k|} (\mathbf{E}[e^{-\beta_0(\omega)}])^{|Q_k|} = e^{-s_0 |Q_k|}, \end{aligned}$$

where we used the independence of the random variables $\{\beta_n\}_{n \in \mathbb{Z}^2}$. \square

The interval between the lowest two eigenvalues of $-\Delta_N^k/2$ is $(\pi/(2k+1))^2/2$. Notice that $\|V_\omega/2\|_\infty \leq 1/2$. By the analytic perturbation theory [31, Theorem 4.1.30], we see that the eigenvalue $E_1(\omega, z)$ can be extended analytically in the region $\{z \in \mathbb{C} \mid |z| < R/2\}$, where

$$R = \left(\frac{\pi}{2k+1} \right)^2.$$

Moreover,

$$|E_1(\omega, z)| < \frac{1}{4} \left(\frac{\pi}{2k+1} \right)^2 \quad (57)$$

for $|z| < R/2$.

Lemma 4.3. Put $C_7 = 2/\pi^2$. Then

$$|E_1(\omega, t) - tE_1'(\omega, 0)| \leq C_7|Q_k|t^2 \quad (58)$$

for every real t with $|t| < R/2$.

Proof. Since $E_1(\omega, 0) = 0$, $E_1(\omega, t) - tE_1'(\omega, 0)$ equals to the remainder term $(E''(\omega, \xi)/2)t^2$ ($|\xi| < |t|$) in the Taylor expansion. By the Cauchy integral formula and (57), we have for $|t| < R/2$ and small $\epsilon > 0$

$$\begin{aligned} \left| \frac{E_1''(\omega, \xi)}{2} \right| &\leq \frac{1}{2\pi} \int_{|z|=(1-\epsilon)R} \left| \frac{E_1(\omega, z)}{(\xi - z)^3} \right| |dz| \\ &\leq \frac{1}{4} \left(\frac{\pi}{2k+1} \right)^2 \frac{(1-\epsilon)R}{((1/2 - \epsilon)R)^3}. \end{aligned}$$

Taking $\epsilon \rightarrow 0$, the right hand side converges to

$$\frac{2}{R^2} \left(\frac{\pi}{2k+1} \right)^2 = C_7|Q_k|.$$

Thus we have the conclusion. \square

Let b be a small positive number, which will be determined later. Assume the event

$$E_1(\omega, 1) \leq \frac{b}{(2k+1)^2}$$

occurs. Since $E_1(\omega, t)$ is monotone non-decreasing with respect to t , we have $E_1(\omega, t) \leq b/(2k+1)^2$ for $|t| \leq 1$. By Lemma 4.3, we have

$$\begin{aligned} E_1'(\omega, 0) &\leq \frac{1}{t} E_1(\omega, t) + C_7(2k+1)^2 t \\ &\leq \frac{b}{t(2k+1)^2} + C_7(2k+1)^2 t \end{aligned} \quad (59)$$

for $|t| < R/2$. The right hand side of (59) takes the minimal value $2\sqrt{bC_7}$ at $t = t_0 = \sqrt{b/C_7}(2k+1)^{-2}$. Take b so small that

$$2\sqrt{bC_7} \leq s_0, \quad t_0 = \sqrt{\frac{b}{C_7}}(2k+1)^{-2} < \frac{R}{2} = \frac{\pi^2}{2}(2k+1)^{-2}.$$

Then we have from (59) with $t = t_0$

$$E'_1(\omega, 0) \leq s_0.$$

This inequality, (55) and (56) implies

$$\begin{aligned} \mathbf{P} \left\{ E_1(\omega, 1) \leq \frac{b}{(2k+1)^2} \right\} &\leq \mathbf{P} \{ E'_1(\omega, 0) \leq s_0 \} \\ &\leq e^{-s_0 |Q_k|}. \end{aligned}$$

This inequality and (53) implies the conclusion of Theorem 1.4.

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